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GENERATION OF THE TOLLMIEN–SCHLICHTING WAVE IN A SUPERSONIC BOUNDARY LAYER BY TWO SINUSOIDAL ACOUSTIC WAVES

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The problem is solved using parabolized equations of stability for three-dimensional perturbations of a compressible boundary layer on a flat plate. Nonlinearity is taken into account by quadratic terms that are most significant in estimates of the viscous critical layer of the stability theory. These terms are determined by the total field of two acoustic perturbations, and the equations become linear and inhomogeneous. The calculations are performed for one acoustic wave being stationary in the reference system fitted to the plate for Mach numbers M = 2 *and* 5*. Solutions are presented, which are identified very accurately with Tollmien–Schlichting waves at a rather large distance from the plate edge.*

In [1], the maximum amplitude of the Tollmien–Schlichting wave over the boundary-layer cross section at the point of the loss of stability is assumed to be equal to the maximum amplitude of a perturbation initiated by sound. Later, attempts were made to justify this assumption (see, for example, [2]). Nevertheless, the author [3], using parabolized stability equations, obtained data indicating that a significant excess of intensity of the perturbation in the boundary layer over the intensity of sound has a resonant character. Different (decaying) modes of eigen perturbations are excited. The phase velocities of sonic and subsonic increasing modes do not coincide by definition. It follows from here that an acoustic wave can excite a Tollmien–Schlichting wave only by interacting with other boundary-layer perturbations.

An action with wave parameters close to parameters of the instability wave may be performed by two acoustic waves through quadratic cross terms of nonlinear equations for perturbations, since the frequencies and wave vectors are summed. In a supersonic boundary layer, one of the acoustic waves may be stationary in the reference system fitted to the plate. This case is of greatest interest, since the stationary perturbation is related to the wetted object and may have a different nature (for instance, surface roughness).

Following [4], we use an orthogonal coordinate system (ξ, ψ, z), where the curves $\psi = \text{const}$ are streamlines of an undisturbed stationary boundary layer (ψ is the stream function) in the plane $z = \text{const}$ perpendicular to the leading edge of the plate, and the coordinate ξ on the wall is the distance to the leading edge. Eliminating terms caused by curvature of the streamlines, which have the order R^{-2} in the case of a flat plate ($R = \sqrt{u_\infty \xi / \nu_\infty}$, the subscript ∞ refers to free-stream parameters), from equations of compressible fluid dynamics, we obtain

$$\operatorname{div} \boldsymbol{v} = e, \qquad d_t \rho + \rho e = 0, \qquad \rho\, d_t v_1 - h_1(\rho v_2^2 - \tau_{22}) = -\partial_1 p + \operatorname{div} \boldsymbol{\tau}_1,$$

$$\rho\, d_t v_2 + h_1(\rho_1 v_1 v_2 - \tau_{12}) = -\partial_2 p + \operatorname{div} \boldsymbol{\tau}_2, \qquad \rho\, d_t v_3 = -\partial_3 p + \operatorname{div} \boldsymbol{\tau}_3,$$

$$\rho\, d_t H = \partial_t p + \operatorname{div} \boldsymbol{q}, \qquad H = h + (v_1^2 + v_2^2 + v_3^2)/2, \qquad \boldsymbol{\tau}_k = (\tau_{k1}, \tau_{k2}, \tau_{k3}),$$

$$q_k = \lambda\, \partial_k T + \boldsymbol{v} \boldsymbol{\tau}_k, \qquad \tau_{kk} = \mu s_{kk} \quad (k = 1, 2, 3), \qquad s_{kk} = 2(\partial_k v_k - e/3) \quad (k = 1, 3),$$

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$$s_{22} = 2(\partial_2 v_2 + h_1 v_1 - e/3), \qquad \tau_{k3} = \mu(\partial_k v_3 + \partial_3 v_k) \quad (k = 1, 2),$$

$$\tau_{12} = \mu(\partial_2 v_1 + \partial_1 v_2 - h_1 v_2), \qquad h_1 = \partial_1 \ln H_2,$$

where $\partial_t = \partial/\partial t$, $\partial_1 = \partial/\partial\xi$, $\partial_2 = (1/H_2)\partial/\partial\psi$, $\partial_3 = \partial/\partial z$, $d_t = \partial_t + \sum_{k=1}^{3} v_k \partial_k$, $\operatorname{div} \boldsymbol{v} = \sum_{k=1}^{3} \partial_k v_k + h_1 v_1$ (for $\boldsymbol{q}$ and $\boldsymbol{\tau}_k$, the expressions are similar), the Lamé coefficient $H_2 = 1/(\rho u)$ is determined by undisturbed stationary parameters, $(v_1, v_2, v_3) = (u, v, w)$ is the velocity vector in the coordinate system (ξ, ψ, z), t is the time, p is the pressure, T is the temperature, h is the enthalpy, ρ is the density, μ is the viscosity, λ is the thermal conductivity, and $\nu = \mu/\rho$.

The quantities u, v, w, p, h, τ_{12}, τ_{23}, and q_2, whose perturbations are used as the basic dependent variables of stability equations in [3], can be represented as the sum $a + \hat{a}$ of boundary-layer and perturbation parameters. The remaining quantities are approximately represented as the sum $a + \hat{a} + \check{a}$, where $\hat{a}$ and $\check{a}$ are linear and quadratic terms with respect to perturbations of the basic quantities, for example, $\check{H} = (\hat{u}^2 + \hat{v}^2 + \hat{w}^2)/2$. Obviously, if different parameters are used as the basic ones (for example, density, temperature, or total enthalpy instead of pressure and enthalpy), the difference in results may be eliminated by reconstruction of third-order and higher terms in equations for perturbations. The quadratic term of the product is written in the form $\check{S}(ab) = \hat{a}\hat{b} + a\check{b} + b\check{a}$. Using the relations $\check{S}(d_t\rho) = \hat{\partial}\hat{\rho} + d\check{\rho}$ and $\check{S}(\rho\, d_t a) = \hat{d}\hat{a} + \hat{\rho}\,\hat{\partial} a + \check{\rho} u\, \partial_1 a + \rho\, d\check{a}$, where

$$\hat{\partial} = \hat{u}\,\partial_1 + \hat{v}\,\partial_2 + \hat{w}\,\partial_3, \qquad \hat{d} = \rho\,\hat{\partial} + \hat{\rho}\, d, \qquad d = \partial_t + u\,\partial_1, \tag{1}$$

we obtain the following equations for perturbations:

$$(\partial_1 + h_1)\hat{u} + \partial_2\hat{v} + \partial_3\hat{w} = \hat{e} + \check{e}, \quad \rho\hat{e} = -(e + d)\hat{\rho} - \hat{\partial}\rho, \quad \rho\check{e} = -(\hat{e} + \hat{\partial})\hat{\rho} - (e + d)\check{\rho},$$

$$(\rho\, d + \hat{d})\hat{u} + (\rho + \hat{\rho})(\hat{u}\,\partial_1 + \hat{v}\,\partial_2)u + (\hat{\rho} + \check{\rho})u\,\partial_1 u - h_1(\rho\hat{v}^2 - \hat{\tau}_{22} - \check{\tau}_{22})$$

$$= -\partial_1\hat{p} + (h_1 + \partial_1)(\hat{\tau}_{11} + \check{\tau}_{11}) + \partial_2\hat{\tau}_{12} + \partial_3(\hat{\tau}_{13} + \check{\tau}_{13}),$$

$$(\rho\, d + \hat{d})\hat{v} + h_1(\rho u\hat{v} + \rho\hat{u}\hat{v} + u\hat{\rho}\hat{v} - \hat{\tau}_{12}) = -\partial_2\hat{p} + (h_1 + \partial_1)\hat{\tau}_{12} + \partial_2(\hat{\tau}_{22} + \check{\tau}_{22}) + \partial_3\hat{\tau}_{23},$$

$$(\rho\, d + \hat{d})\hat{w} = -\partial_3\hat{p} + (h_1 + \partial_1)(\hat{\tau}_{13} + \check{\tau}_{13}) + \partial_2\hat{\tau}_{23} + \partial_3(\hat{\tau}_{33} + \check{\tau}_{33}),$$

$$(\rho\, d + \hat{d})\hat{H} + (\rho + \hat{\rho})(\hat{u}\,\partial_1 + \hat{v}\,\partial_2)H + (\hat{\rho} + \check{\rho})u\,\partial_1 H + \rho\, d\check{H}$$

$$= \partial_t\hat{p} + (h_1 + \partial_1)(\hat{q}_1 + \check{q}_1) + \partial_2\hat{q}_2 + \partial_3(\hat{q}_3 + \check{q}_3),$$

$$\hat{\tau}_{12} = (\mu + \hat{\mu})(\partial_2\hat{u} + \partial_1\hat{v} - h_1\hat{v}) + (\hat{\mu} + \check{\mu})\,\partial_2 u, \qquad \hat{\tau}_{23} = (\mu + \hat{\mu})(\partial_2\hat{w} + \partial_3\hat{v}), \tag{2}$$

$$\hat{q}_2 = (\lambda + \hat{\lambda})\,\partial_2\hat{T} + (\hat{\lambda} + \check{\lambda})\,\partial_2 T + \lambda\,\partial_2\check{T} + u\hat{\tau}_{12} + \tau_{12}\hat{u} + \tau_{22}\hat{v} + \hat{\tau}_{12}\hat{u} + \hat{\tau}_{22}\hat{v} + \hat{\tau}_{23}\hat{w},$$

$$\hat{\tau}_{13} = \mu(\partial_1\hat{w} + \partial_3\hat{u}), \quad \check{\tau}_{13} = \hat{\mu}\hat{\tau}_{13}/\mu, \quad \hat{\tau}_{kk} = \mu\hat{s}_{kk} + s_{kk}\hat{\mu}, \quad \check{\tau}_{kk} = \mu\check{s}_{kk} + s_{kk}\check{\mu} + \hat{\mu}\hat{s}_{kk},$$

$$\hat{s}_{11} = 2(\partial_1\hat{u} - \hat{e}/3), \qquad \hat{s}_{22} = 2(h_1\hat{u} + \partial_2\hat{v} - \hat{e}/3),$$

$$\hat{s}_{33} = 2(\partial_3\hat{w} - \hat{e}/3), \qquad \check{s}_{11} = \check{s}_{22} = \check{s}_{33} = -2\check{e}/3,$$

$$\hat{q}_1 = \lambda\,\partial_1\hat{T} + \hat{\lambda}\,\partial_1 T + u\hat{\tau}_{11} + \tau_{11}\hat{u} + \tau_{12}\hat{v}, \quad \hat{q}_3 = \lambda\,\partial_3\hat{T} + u\hat{\tau}_{13} + \tau_{33}\hat{w},$$

$$\check{q}_1 = \check{\lambda}\,\partial_1 T + \lambda\,\partial_1\check{T} + \hat{\lambda}\,\partial_1\hat{T} + u\check{\tau}_{11} + \hat{\tau}_{11}\hat{u} + \hat{\tau}_{12}\hat{v} + \hat{\tau}_{13}\hat{w},$$

$$\check{q}_3 = \lambda\,\partial_3\check{T} + \hat{\lambda}\,\partial_3\hat{T} + u\check{\tau}_{13} + \hat{\tau}_{13}\hat{u} + \hat{\tau}_{23}\hat{v} + \hat{\tau}_{33}\hat{w}, \quad e = (h_1 + \partial_1)u, \quad \tau_{12} = \mu\,\partial_2 u,$$

$$s_{11} = 2(\partial_1 u - e/3), \quad s_{22} = 2(h_1 u - e/3), \quad s_{33} = -2e/3.$$

The Navier–Stokes equations that are not supplemented by equations of state are not changed by introducing the following scales: ν_∞/u_∞^2 for time, ν_∞/u_∞ for length, μ_∞ for the stream function, $\rho_\infty u_\infty^2$ for pressure and viscous stresses, u_∞^2 for enthalpy, $\mu_\infty u_\infty^2/T_\infty$ for thermal conductivity, and $\rho_\infty u_\infty^3$ for the heat flux. The remaining quantities are normalized to their free-stream values. In the scales used, as $R \to \infty$, the thickness of the critical layer in order of magnitude is $R^{2/3}$ (rather than $R^{-1/3}$, as in [4], where it was estimated with respect to the boundary-layer thickness), $\partial_t, \partial_1 = O(R^{-1})$, $\partial_2 = O(R^{-2/3})$, $\hat{p}, \hat{v} = O(1)$, $\hat{u}, \hat{w}, \hat{h} = O(R^{1/3})$, $d = O(R^{-4/3})$, and $\hat{\tau}_{12}, \hat{\tau}_{23}, \hat{q}_2 = O(R^{-1/3})$; for the main flow parameters, we have $\partial_1 = O(R^{-2})$ and $\partial_2 = O(R^{-1})$. The estimate $\hat{p}$ is assumed to be the same as that outside the critical layer due to continuity of pressure perturbations at the critical point in equations of the inviscid stability theory.

Rejecting in (2) nonlinear terms of order $R^{-1/3}$ with respect to the main linear terms, we obtain

$$\partial_2 \hat{v} - \hat{L}_v = -(1/\rho)[\hat{\partial}\hat{\rho} - (\hat{\rho}/\rho)(d\hat{\rho} + \hat{v}\,\partial_2\rho) + d\check{\rho}],$$

$$\partial_2(\hat{p} - \hat{\tau}_{22}) - \hat{L}_p = -\rho\,\hat{\partial}\hat{v}, \qquad \partial_2\hat{\tau}_{12} - \hat{L}_{12} = \hat{d}\hat{u} + \hat{\rho}\hat{v}\,\partial_2 u, \qquad \partial_2\hat{\tau}_{23} - \hat{L}_{23} = \hat{d}\hat{w},$$

$$\partial_2\hat{q} - \hat{L}_q = \hat{d}\hat{h} + u\,\hat{d}\hat{u} + \rho\hat{u}\hat{v}\,\partial_2 u + \hat{\rho}\hat{v}\,\partial_2 H + \rho(\hat{u}\,d\hat{u} + \hat{w}\,d\hat{w}), \qquad (3)$$

$$\partial_2\hat{u} - \hat{L}_u = -(\hat{\mu}/\mu)\,\partial_2\hat{u} - (\check{\mu}/\mu)\,\partial_2 u, \qquad \partial_2\hat{w} - \hat{L}_w = -(\hat{\mu}/\mu)\,\partial_2\hat{w},$$

$$\partial_2\hat{T} - \hat{L}_T = -(\hat{\lambda}/\lambda)\,\partial_2\hat{T} - \partial_2\check{T} - (\check{\lambda}/\lambda)\,\partial_2 T - (\hat{u}\hat{\tau}_{12} + \hat{w}\hat{\tau}_{23})/\lambda,$$

where $\hat{q} = \hat{q}_2$; the operators $\hat{\partial}$, $\hat{d}$, and d are defined in (1); $\hat{L}_v$, $\hat{L}_p$, ... are linear expressions that do not contain ∂_2 and are given below in a parabolized form. In the case of a perfect gas with a constant Prandtl number Pr, the latter equation is transformed to

$$\partial_2\hat{h} - \hat{L}_h = -\mu_h\hat{h}\,\partial_2\hat{h} - \mu_{hh}\hat{h}^2\,\partial_2 h - \mathrm{Pr}\,(\hat{u}\hat{\tau}_{12} + \hat{w}\hat{\tau}_{23})/\mu$$

and $\hat{\mu}/\mu = \mu_h\hat{h}$, $\check{\mu}/\mu = \mu_{hh}\hat{h}^2$, $\mu_h = d\ln\mu/dh$, $\mu_{hh} = (d^2\mu/dh^2)/(2\mu)$, $\check{T} = 0$, and $\check{\rho} = \hat{\rho}\hat{h}/h$.

System (3) with regard for the equations of state and laws of viscosity and heat conductivity of the gas has the form $K\hat{\boldsymbol{Z}} = \hat{N}(\hat{\boldsymbol{Z}})$, where $\hat{\boldsymbol{Z}} = (\hat{v}, \hat{p}, \hat{\tau}_{12}, \hat{\tau}_{23}, \hat{q}, \hat{u}, \hat{w}, \hat{h})$ is the vector composed of perturbations of the main parameters and K and $\hat{N}$ are the linear (matrix) and nonlinear differential operators, respectively. The solution of system (3) is sought as the sum $\hat{\boldsymbol{Z}} + \hat{\boldsymbol{Z}}_1 + \hat{\boldsymbol{Z}}_2$, where $\hat{\boldsymbol{Z}}_1$ and $\hat{\boldsymbol{Z}}_2$ are linear perturbations initiated by sound (see [3]). If $\hat{\boldsymbol{Z}}$ is an infinitesimal of higher order than $\hat{\boldsymbol{Z}}_1$ and $\hat{\boldsymbol{Z}}_2$, it obeys the equation

$$K\hat{\boldsymbol{Z}} = \hat{N}(\hat{\boldsymbol{Z}}_1 + \hat{\boldsymbol{Z}}_2), \qquad (4)$$

i.e., nonlinear equations become inhomogeneous linear ones.

For acoustic waves of the form $\hat{\boldsymbol{Z}}_j = \mathrm{Re}\,[\tilde{\boldsymbol{Z}}_j \exp(i\varphi_j)]$ ($\varphi_j = \alpha_j\xi + \beta_j z - \omega_j t$ and Re is the real part), the amplitude functions are determined by the equations $L\tilde{\boldsymbol{Z}}_j = 0$. The right side of Eq. (4) consists of the products of components of the vectors $\tilde{\boldsymbol{Z}}_j$ and $\tilde{\boldsymbol{Z}}_k$: $\hat{a}_j\hat{b}_k = \mathrm{Re}\,\{\tilde{a}_j\tilde{b}_k \exp[i(\varphi_j + \varphi_k)] + \tilde{a}_j\tilde{b}_k^* \exp[i(\varphi_j - \varphi_k)]\}$. In the present work, we are interested in cross combinations $j \neq k$, and $\hat{a}\hat{b} = \mathrm{Re}\,\{(\tilde{a}_1\tilde{b}_2 + \tilde{a}_2\tilde{b}_1)\exp[i(\varphi_1 + \varphi_2)]\}$.

The solution of Eq. (4) of the form $\hat{\boldsymbol{Z}} = \mathrm{Re}\,\{\tilde{\boldsymbol{Z}} \exp[i(\int \alpha\,d\xi + \beta z - \omega t)]\}$, $\omega = \omega_1 + \omega_2$, $\beta = \beta_1 + \beta_2$ is constructed by solving the equation

$$L\tilde{\boldsymbol{Z}} = \boldsymbol{N} \exp\Big[i\int(\alpha_3 - \alpha)\,d\xi\Big], \qquad \alpha_3 = \alpha_1 + \alpha_2, \qquad (5)$$

where L and $\boldsymbol{N}$ are, respectively, the operator and vector corresponding to the left and right sides of system (3) with the substitutions $\partial_t \to -i\omega$, $\partial_1 \to i\alpha + \partial_1$, $\partial_3 \to i\beta$, and $\hat{a}\hat{b} \to \tilde{a}_1\tilde{b}_2 + \tilde{a}_2\tilde{b}_1 = \tilde{a}_j\tilde{b}_k$ (the sign of summation is omitted). The right sides in the approximation used, with regard for the estimate $\partial_1 = O(R^{-5/3})$ (the estimate differs from that given previously, since the operator ∂_1 acts here on the amplitude functions $\tilde{a}$ rather than on the perturbation $\hat{a}$ itself), take the form

$$N_v = -\tilde{D}_j\tilde{r}_k + \rho\tilde{v}_j(\partial_2\tilde{T}_k - \rho\tilde{T}_k\,\partial_2 T) + 3\rho^2 u_{cj}\tilde{T}_j\tilde{T}_k, \qquad N_p = -\rho(\tilde{D}_j + \tilde{v}_j\,\partial_2)\tilde{v}_k,$$

$$N_{12} = \tilde{d}_j\tilde{u}_k + \rho\tilde{v}_j\tilde{r}_k\,\partial_2 u, \qquad N_{23} = \tilde{d}_j\tilde{w}_k,$$

$$N_q = \tilde{d}_j\tilde{h}_k + u\,\tilde{d}_j\tilde{u}_k + \rho[\tilde{v}_j(\tilde{u}_k\,\partial_2 u + \tilde{r}_k\,\partial_2 H) + \tilde{u}_j(u_{cj} + \partial_1)\tilde{u}_k + \tilde{w}_j(u_{cj} + \partial_1)\tilde{w}_k], \qquad (6)$$

$$N_u = -\mu_h\tilde{h}_j\,\partial_2\tilde{u}_k - \mu_{hh}\tilde{h}_j\tilde{h}_k\,\partial_2 u, \qquad N_w = -\mu_h\tilde{h}_j\,\partial_2\tilde{w}_k,$$

$$N_h = -\mu_h\tilde{h}_j\,\partial_2\tilde{h}_k - \mu_{hh}\tilde{h}_j\tilde{h}_k\,\partial_2 h - \mathrm{Pr}\,(\tilde{u}_j\tilde{\tau}_{12k} + \tilde{w}_j\tilde{\tau}_{23k})/\mu,$$

where $\tilde{D}_j = i(\alpha_3 - \alpha_j)\tilde{u}_j + i(\beta - \beta_j)\tilde{w}_j$, $u_{cj} = i\alpha_j u - i\omega_j$, $\tilde{d}_j = \rho[\tilde{D}_j + (u_{c3} - u_{cj})\tilde{r}_j + (\tilde{u}_j + u\tilde{r}_j)\partial_1 + \tilde{v}_j\partial_2]$, and $\tilde{r} = \tilde{\rho}/\rho$.

The transition to the similarity variable of the boundary layer η is performed using the formulas $\partial_1 = R^{-1}(\partial + f_1\,\partial_\eta)$ and $\partial_2 = R^{-1}\rho\,\partial_\eta$, where $R = \sqrt{\xi}$, $f_1 = -f_0 u$, $f_0 = f/(2Ru^2)$, $f = \psi/R$, $\partial = 0.5\partial/\partial R$, and $\partial_\eta = \partial/\partial\eta$. The calculations [3] showed that the maximum of the amplitude of the forced perturbation is located near the coordinate line $\eta = \text{const}$; hence, we may consider that $\partial = O(R^{-1})$ and ignore this derivative in the substitution of variables in system (6). As the homogeneous linear part of the transformed equations (5), we use the equations of [3] parabolized with the help of estimates in terms of the integer powers of R. In the calculations, we reject the term containing $\partial\tilde{v}/\partial R$, which allows us to reduce the step of the marching scheme of integration due to an insignificant decrease in accuracy from $R^{-5/3}$ to $R^{-4/3}$. The final equations have the following form:

$$\begin{gathered}
\tilde{v}' = \tilde{v}^* - g_m uT\,\partial\tilde{\pi} - T\,\partial\tilde{u} + g_{m1}u\,\partial\tilde{h} - g_m f_2 T\tilde{\pi}^* - f_3\tilde{u}^* + g_{m1}f_2\tilde{h}^* \\
- \Sigma[\tilde{D}_j\tilde{r}_k + (h'\tilde{v}_j - 3c_j\tilde{h}_j)\tilde{h}_k/h^2 - \tilde{v}_j\tilde{h}'_k/h], \\
\tilde{\pi}' = \tilde{\pi}^* - u\,\partial\tilde{v} - f_2\tilde{v}^* - \Sigma\rho(\tilde{D}_j + \tilde{v}'_j)\tilde{v}_k, \\
\tilde{\tau}'_{12} = \rho u'\tilde{v} + T\,\partial\tilde{\pi} + (i_c + f_1 u' + u\,\partial)\tilde{u} + i_x\tilde{p} + f_2 u'\tilde{r} - \tilde{i}_T + f_3\tilde{\pi}^* + f_2\tilde{u}^* + \Sigma(\tilde{b}_{1jk} + \rho u'\tilde{b}_{2jk}), \\
\tilde{\tau}'_{23} = (i_c + u\,\partial)\tilde{w} + i_z\tilde{p} - i_x\tilde{\tau}_{13} - i_z\tilde{\tau}_{33} + f_2\tilde{w}^* + \Sigma(\tilde{d}_j\tilde{w}_k + \tilde{b}_j\tilde{w}'_k), \\
\tilde{u}' = \tilde{u}^* + \Sigma(\tilde{\mu}_j\tilde{u}'_k + \tilde{\mu}_{jk}u'), \qquad \tilde{w}' = \tilde{w}^* + \Sigma\tilde{\mu}_j\tilde{w}'_k, \qquad (7) \\
\tilde{q}' = (\rho H' - \mu_x u')\tilde{v} + (i_c u + f_1 H' + f_2 u' + u^2\,\partial)\tilde{u} \\
+ [i_c - (i_x^2 + i_z^2)\mu_R/\mathrm{Pr} + u\,\partial]\tilde{h} + i_w\tilde{p} + f_2 H'\tilde{r} - u\tilde{i}_T + f_2(u\tilde{u}^* + \tilde{h}^*) \\
+ \Sigma\{\tilde{d}_j\tilde{h}_k + \tilde{b}_j\tilde{h}'_k + u\tilde{b}_{1jk} + \rho[u'\tilde{v}_j\tilde{u}_k + H'\tilde{b}_{2jk} + c_j(\tilde{u}_j\tilde{u}_k + \tilde{w}_j\tilde{w}_k) + f_3(\tilde{u}_j\tilde{u}'_k + \tilde{w}\tilde{w}'_k)]\}, \\
\tilde{h}' = \tilde{h}^* + \Sigma[\tilde{\mu}_j\tilde{h}'_k + \tilde{\mu}_{jk}h' - \mathrm{Pr}\,(\tilde{u}_j\tilde{\tau}_{12k} + \tilde{w}_j\tilde{\tau}_{23k})/\mu_R].
\end{gathered}$$

Here $\tilde{v}^* = \rho T'\tilde{v} - (i_x + f_0 u'T)\tilde{u} - i_z\tilde{w} - f_2\rho T'\tilde{T} - i_c T\tilde{r}$, $\tilde{u}^* = -i_x\tilde{v} + \tilde{\tau}_{12}/\mu_R - u'\mu_h\tilde{h}$, $\tilde{r} = \tilde{\rho}/\rho = g_m\tilde{p} - \rho\tilde{T}$, $\tilde{\pi}^* = (f_1 u' - f_2\rho T' - i_c)\tilde{v} + i_x\tilde{\tau}_{12} + i_z\tilde{\tau}_{23}$, $\tilde{w}^* = -i_z\tilde{v} + \tilde{\tau}_{23}/\mu_R$, $\tilde{h}^* = -\mathrm{Pr}\,u'\tilde{u} - h'\mu_h\tilde{h} + \mathrm{Pr}\,(\tilde{q} - u\tilde{\tau}_{12})/\mu_R$, $\tilde{p} = \tilde{\pi} - \tilde{\tau}_{11} - \tilde{\tau}_{33}$, $\tilde{i}_T = i_x\tilde{\tau}_{11} + i_z\tilde{\tau}_{13}$, $\tilde{\tau}_{11} = 2\mu_x\tilde{u} - \tilde{e}_3$, $\tilde{\tau}_{33} = 2\mu_z\tilde{w} - \tilde{e}_3$, $\tilde{\tau}_{13} = \mu_x\tilde{w} + \mu_z\tilde{u}$, $\tilde{e}_3 = 2\mu_R[\rho T'\tilde{v} - i_c(g_m T\tilde{\pi} - \tilde{T})]/3$, $\tilde{T} = g_{m1}\tilde{h}$, $\tilde{D}_j = (i_{x3} - i_{xj})\tilde{u}_j + (i_z - i_{zj})\tilde{w}_j$, $\tilde{d}_j = \rho[\tilde{D}_j + (c_3 - c_j)\tilde{r}_j]$, $\tilde{b}_{1jk} = \tilde{d}_j\tilde{u}_k + \tilde{b}_j\tilde{u}'_k$, $\tilde{b}_{2jk} = \tilde{v}_j\tilde{r}_k$, $\tilde{b}_j = \tilde{v}_j + f_3(\tilde{u}_j + u\tilde{r}_j)$ $\tilde{\mu}_j = -\mu_h\tilde{h}_j$, $\tilde{\mu}_{jk} = -\mu_{hh}\tilde{h}_j\tilde{h}_k$, $\mu_x = i_x\mu_R$, $\mu_z = i_z\mu_R$, $\mu_R = \mu\rho/R$, $\mu_h = g_{m1}d\ln\mu/dT$, $\mu_{hh} = g_{m1}^2(d^2\mu/dT^2)/(2\mu)$, $c_j = i_{xj}u - i_{wj}$, $i_c = iR(u\alpha - \omega)$, $i_x = i\alpha RT$, $i_z = i\beta RT$, $i_w = i\omega RT$, $f_2 = f_1 u$, $f_3 = f_1 T$, $\Sigma = \exp\Big[2i\displaystyle\int(\alpha_3 - \alpha)R\,dR\Big]\sum_{j=1,\,k=3-j}^{2}$, $g_m = \gamma\mathrm{M}^2$, $g_{m1} = (\gamma - 1)\mathrm{M}^2$, M is the Mach number, γ is the ratio of specific heats, and $\partial = 0.5\partial/\partial R$; the prime denotes the derivative with respect to η.

The inhomogeneous parts of Eqs. (7) in the critical layer are $R^{2/3}$ times greater in order of magnitude than the estimate for the region outside the critical layer, i.e., the sought perturbation is generated in the critical layer of the perturbation initiated by sound. Outside the boundary layer, the inhomogeneous terms may be neglected; then the problem is solved with the same boundary conditions and by the same method as the problem of stability for parabolized equations [4].

The solution of the problem for ordinary differential equations $\boldsymbol{Z}' = A\boldsymbol{Z} + B\boldsymbol{Z}_0 + \boldsymbol{C}(\boldsymbol{Z}_1, \boldsymbol{Z}_2)$ obtained by the approximation $\partial\boldsymbol{Z}/\partial R = (\boldsymbol{Z} - \boldsymbol{Z}_0)/(R - R_0)$ is constructed as a superposition of the solution calculated under the condition $Z = 0$ at the boundary-layer edge and four fundamental solutions of the homogeneous equations $\boldsymbol{Z}' = A\boldsymbol{Z}$ under conditions of decay as $\eta \to \infty$ and with coefficients determined by the conditions $\tilde{u} = \tilde{v} = \tilde{w} = \tilde{h} = 0$ on the wall. The value of α at each step in R is determined by Newton's method if the requirement $\partial\tilde{\pi}/\partial R = 0$ is satisfied at the point of the maximum amplitude A of the mass-flow perturbation $\tilde{m} = \rho\tilde{u} + u\tilde{\rho}$ (see [4]). In the initial cross section of the boundary layer, $\boldsymbol{Z}$ is calculated for $\alpha = \alpha_3$ using local equations obtained by elimination of all derivatives with respect to R from Eqs. (7).

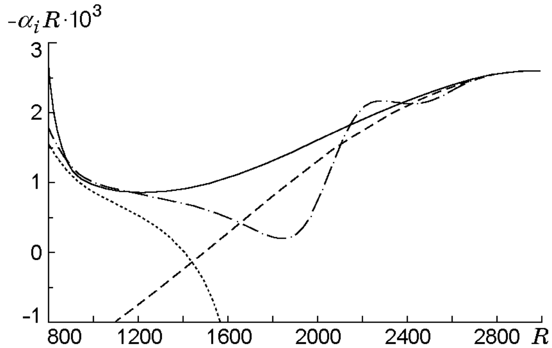

Fig. 1

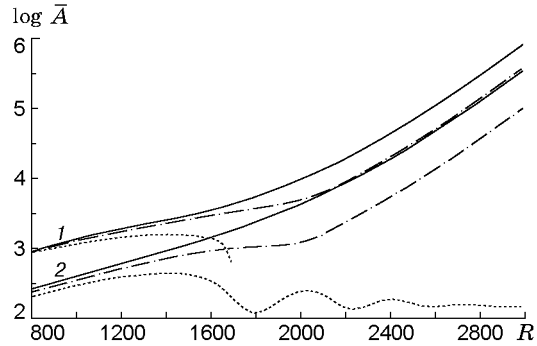

Fig. 2

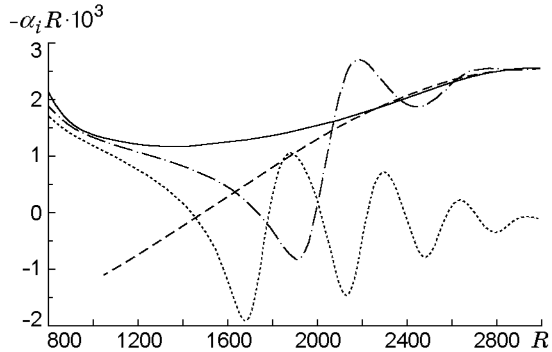

Fig. 3

The calculations are performed for a heat-insulated plate for M = 2 and 5. The results are represented as dependences of the maximum relative amplitude of the mass flow $\bar{A} = A/(A_1 A_2)$ over the boundary-layer cross section on R (A_1 and A_2 are the amplitudes of perturbations of the mass flow of the incident acoustic waves).

For M = 2, we use the Sutherland viscosity formula with constant $T_s = 110.4$ K and Pr = 0.71. The excited frequency is assumed to be equal to $\omega = \omega_1 = 4 \cdot 10^{-6}$, and the z-component of the wavenumber is $\beta = \beta_1 + \beta_2 = 2 \cdot 10^{-5}$. The Tollmien–Schlichting wave with the same parameters is directed at an angle of approximately 60° to the flow direction.

Figures 1 and 2 (curves 1) show the results obtained for $\alpha_1 = 3.1 \cdot 10^{-5}$ and $\beta_1 = 3.7 \cdot 10^{-5}$, which corresponds to the angle $\varphi_1 = 50°$ [hereinafter, $\varphi = \arctan(\beta/\alpha)$]. The growth rates $\alpha_i = \operatorname{Im}\alpha$ of forced perturbations in Fig. 1 and their amplitudes in Fig. 2 are calculated for different values of α_2 for steady waves: the solid curves refer to $\alpha_2 = -2.1 \cdot 10^{-5}$, the dot-and-dashed curves to $\alpha_2 = -1.9 \cdot 10^{-5}$, and the dotted curves to $\alpha_2 = -1.8 \cdot 10^{-5}$. The dashed curve in Fig. 1 shows the growth rates of the Tollmien–Schlichting wave. For rather high values of R, within the range $\alpha_2 = (-1.9 \cdot 10^{-5})$–$(-2.1 \cdot 10^{-5})$ ($\varphi_2 = 42$–$39°$), forced oscillations are almost indistinguishable from the growing eigen perturbation. Outside this range, the Tollmien–Schlichting wave is not excited.

Similar results obtained for $\alpha_1 = 1.01 \cdot 10^{-6}$ and $\beta_1 = 5.73 \cdot 10^{-6}$ ($\varphi_1 = 80°$) are plotted in Fig. 2 (curves 2) and Fig. 3. The solid curves refer to $\alpha_2 = 9.5 \cdot 10^{-6}$, dot-and-dashed curves to $\alpha_2 = 1.1 \cdot 10^{-5}$, and dotted curves to $\alpha_2 = 1.2 \cdot 10^{-5}$. The instability wave shown by the dashed curve in Fig. 3 (the same as in Fig. 1) is excited in the range of wavenumbers of the stationary external perturbation $\alpha_2 = (9.5 \cdot 10^{-6})$–$(1.1 \cdot 10^{-5})$ ($\varphi_2 = 56$–$52°$).

The parameters of forced perturbations are rather different from the parameters of the Tollmien–Schlichting waves at the point of the loss of stability $R^* = 1480$; the coincidence is observed only for $R > 3000$. The effective value of the amplitude of the excited eigen perturbation at the point of the loss of stability (receptivity coefficient) is determined by the formula $\bar{A}^* = \bar{A}/K_s$, where $K_s = (A/A^*)_s$ is the coefficient of spatial growth of the Tollmien–Schlichting wave, which is also calculated as $\bar{A}$, for $R = 3000$. The solid curves 1 and 2 in Fig. 2 refer to $\bar{A}^* = 5140$ and 2160, respectively.

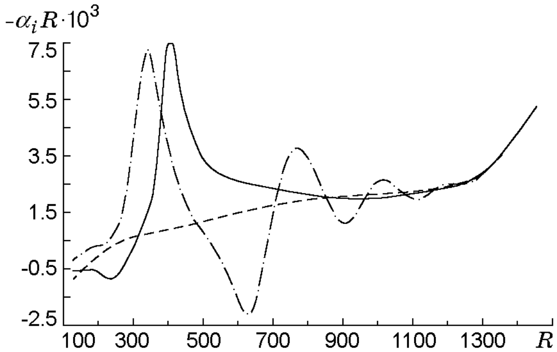

Fig. 4

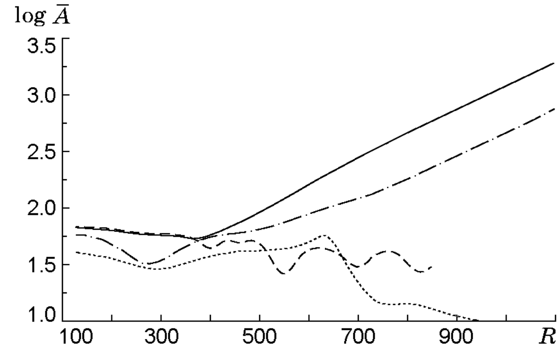

Fig. 5

For M = 5, the calculations were performed for Pr = 1 and $\mu = T$. Two-dimensional perturbations $\beta_1 = \beta_2 = 0$, $\omega_1 = 10^{-4}$, and $\alpha_1 = 5 \cdot 10^{-5}$ were considered. The solid curves in Figs. 4 and 5 show the results obtained for $\alpha_2 = 6.6 \cdot 10^{-5}$, and the dot-and-dashed curves refer to $\alpha_2 = 7.6 \cdot 10^{-5}$. It is seen in Fig. 4 that the parameters of forced perturbations in both cases coincide with the Tollmien–Schlichting wave parameters shown by the dashed curve, beginning from $R \approx 1300$. The above values of α_2 correspond to the receptivity coefficients $\bar{A}^* = 89$ and 34. For $\alpha_2 = 6.5 \cdot 10^{-5}$ (dashed curve in Fig. 5) and $\alpha_2 = 7.7 \cdot 10^{-5}$ (dotted curve), instability waves are not excited, and perturbation amplitudes do not increase at high values of R.

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